SM223 – Calculus III with Optimization | Fall 2017
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Lesson 5. The Cross Product

1 Today...

- Computing the cross product
- The right-hand rule
- Finding areas with the cross product

2 Computing the cross product

- If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\vec{a}$ and $\vec{b}$ is

- Note: $\vec{a} \times \vec{b}$ is a <u>vector</u> (unlike the dot product)
- The cross product is only defined for 3D vectors
- Mnemonic for taking the cross product:

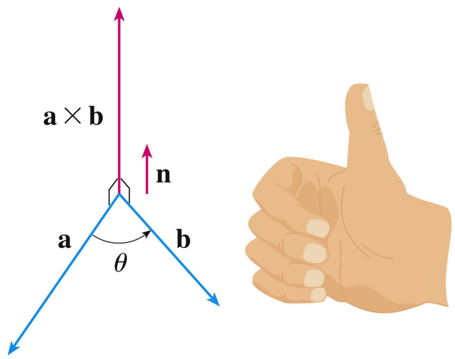
$$\begin{array}{cccccc}
+ & + & + & - & - & - \\
\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\
a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 & b_1 & b_2 & b_3
\end{array}$$

Example 1. Let $\vec{a} = \langle 1, 3, 4 \rangle$ and $\vec{b} = \langle 2, 7, -5 \rangle$. Find $\vec{a} \times \vec{b}$.

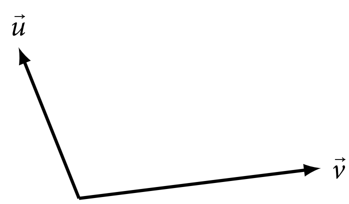
3 The right-hand rule

- The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.
- Orthogonal which way? **Right-hand rule**
  - Curl fingers of right hand from $\vec{a}$ to $\vec{b}$

  $\Rightarrow$ Thumb points in direction of $\vec{a} \times \vec{b}$



Example 2. Find the direction of $\vec{u} \times \vec{v}$.



Example 3. Find two unit vectors orthogonal to both $\vec{a} = 2\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} + 4\vec{j}$.

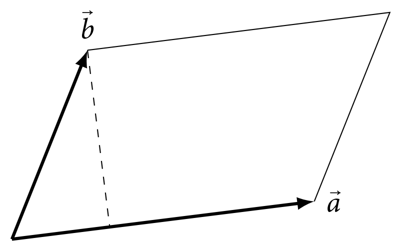
4 Areas and the cross product

- What about the magnitude of $\vec{a} \times \vec{b}$?
- If θ is the angle between $\vec{a}$ and $\vec{b}$, then

- $\sin\theta = 0$ when $\theta =$

$\Rightarrow$ Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

- $|\vec{a} \times \vec{b}|$ = the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$:



Example 4. Find the area of the triangle with vertices $P(1,4,2)$, $Q(-2,5,-1)$, and $R(1,3,1)$.

- Cross products between $\vec{i}$, $\vec{j}$ and $\vec{k}$ are pretty easy to remember:

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$
$$\vec{j} \times \vec{i} = -\vec{k} \qquad \vec{k} \times \vec{j} = -\vec{i} \qquad \vec{i} \times \vec{k} = -\vec{j}$$

  - Mnemonic:

$$\begin{tikzcd} i \arrow[rr] & & j \arrow[dl] \\ & k \arrow[ul] & \end{tikzcd}$$

- **Properties of cross products:** if $\vec{a}, \vec{b}, \vec{c}$ are vectors and c is a scalar:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$
$$(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b}) \qquad \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$
$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \qquad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

- The cross product <u>is not commutative</u>, i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$
- The cross product <u>is not associative</u> either, i.e. $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$